IB Mathematics Analysis and Approaches HL

Internal Assessment

How can the shape of a swimmer's dolphin kick be modeled and optimized?

- Investigating the patterns of swimming through mathematics -

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20 pages

1 Intro

The dolphin kick is a cornerstone of competitive swimming, a technique present in all swimming strokes. It combines athleticism with grace, with its name derived from the undulating motion that mimics the movements of dolphins. Dolphin kicks revolutionized the scene of swimming, introduced in the 1970s by a swimmer named Jesse Vassallo using it underwater, and its prominence slowly ramped up (Mortenson, 2023). When the Seoul 1988 olympics hit, the underwater dolphin kick was dare say exploited, with backstrokers swimming around two thirds of the entire pool underwater. This led to a restriction where swimmers can swim no further than 15 metres underwater in 1991, showing how much of a powerhouse the underwater dolphin kick is.

When I started competitively swimming, I was introduced to the underwater dolphin kick. My coach put a ton of emphasis on these kicks, making us do at least two or three off the wall before surfacing. This motivated me to try and find the optimal dolphin kick form, as in, how I could travel the furthest possible distance underwater as quickly as possible, with the least amount of drag. I soon realised the motion of dolphin kicks generally resembled the shape of sinusoidal functions. Thus, the research question was developed:

How can the shape of a swimmer's dolphin kick be modeled and optimized?

This exploration is going to gather and extract data from a video of myself performing dolphin kicks through the use of motion tracking software. The data, in the form of points, will be plotted onto a graph to take a look at its shape. In order to optimize its shape, I am going to examine both the amplitude and frequency of the kicks. I will develop an equation with velocity as its parameter that balances thrust and drag. Using derivatives, the critical points can be found and, hypothetically, the optimal velocity will be determined.

2 Background Information and Hypothesis

This mathematical exploration is going to be based on the concept of the 'function.' There are many different types of functions: linear, quadratic, polynomial, and sinusoidal to name a few. These functions each have their own properties, and so if I want to create a mathematical model of dolphin kicks, I would need to account for these differences. It is possible that the dolphin kick cannot be modeled with one function type and it would need a combination of these different functions to model its shape.

Before collecting data, a possible prediction could be that the shape of the dolphin kick can be represented and graphed using a sinusoidal function. This sinusoidal function would be in the form:

$$f(t) = A\sin(Bt + C) + D$$

where A, B, C and D are constants to be found.

Additionally, it is important to note how the independent variable of this function, t, represents time, as it would in the real world.

3 Methodology and Data Collection

After establishing an initial hypothesis, in order to actually model the shape of a swimmer's dolphin kick, I would need footage of someone performing them. This is the first step in the methodology. After consideration, I decided that I would be using footage of myself performing dolphin kicks in order to create a model. That way, I could record multiple different trials and not have to use footage from the internet.

i. Setting up for Data Collection

The first step to gather data is to establish what equipment would be needed, as well as any considerations I would need to make. Immediately, the first problem raises the question of how I would record underwater footage. I decided to use a GoPro as their cameras have underwater cases made to record in wet conditions. After figuring out how to set up the GoPro, the next step would be to consider the dolphin kick itself. I debated between either starting from a dive or starting by pushing off the wall, however I chose the latter because I was concerned that the splash caused by the dive would hinder the vision of my ankle. This is important because I would need to track a certain part of my body to gather points, and I chose my ankle because it is the most stable part of my body and has enough movement to create a sophisticated model.

ii. Variables

From this stage, we are now able to define our independent and dependent variables, as well as any control variables that may be needed when gathering data. The dependent variable would be the position of the ankle, and the independent variable would be time. Some important control variables are the position and how the GoPro is angled in the water. Any minor shifts could skew any different trials. I would also need to try and keep the amplitude and frequency of my kicks as consistent as I can achieve if I want to create a sinusoidal graph.

iii. Motion Tracking

After recording the trials on the GoPro and transferring those files over to my computer, the next concern would be to track the motion of my ankle in order to create points for my graph.

Without having any tracking on my ankle, the footage would be effectively useless, as there would be no way to have a reference point for each frame of the video. Manually tracking the motion of my ankle would be incredibly difficult and tedious, so I resorted to online software to perform this motion tracking automatically. This software is called Adobe After Effects (Adobe, 2023). I used their motion tracking tool in order to gather points. The motion tracking began as soon as my ankle entered the shot, and ended when my foot left the frame. This tracking took place over 72 frames, leaving one point every frame, and the video was recorded at 29.97 frames per second (fps). This means that the tracking clip ran for approximately 2.40 seconds.

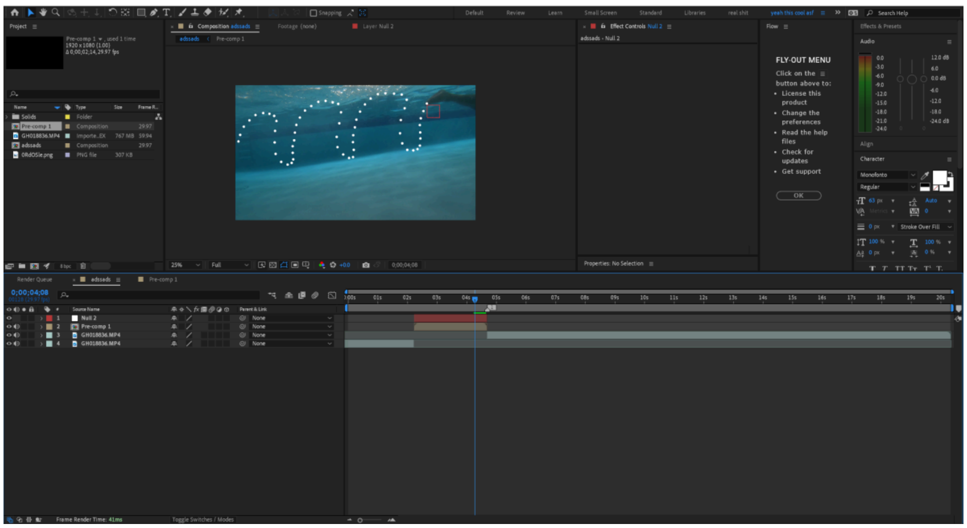


Figure 1. Adobe After Effects tracking the motion of my dolphin kicks (Author's Own, 2025)

iv. The Data

After running the video clip through After Effects, the data has been gathered. Eliminating the other trials were simple, as most of them created a splash and the motion tracking did not capture the points. For some others, I also resurfaced too early. The final data gathered looks like:

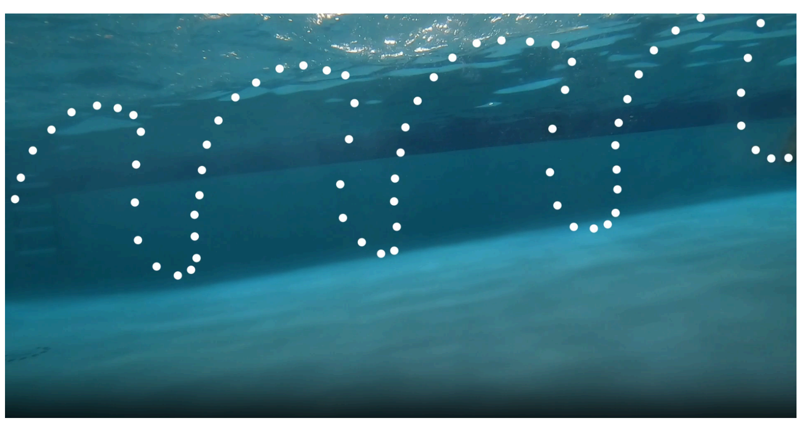


Figure 2. Data produced by solely the motion tracking (Author's Own, 2025)

I traced these dots using an online drawing tool. This would be useful in visualising the general shape of these points.

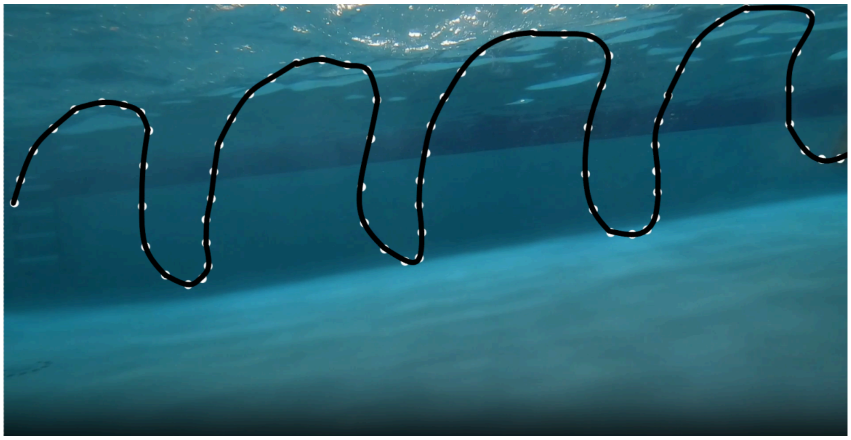


Figure 3. Traced data points (Author's Own, 2025)

As shown, these points do not resemble a function after a curve is drawn through them. What this means is that the original hypothesis would not work. This relation can not be modelled using a singular sinusoidal function. Therefore, I need to use a piecewise function to model the shape of my dolphin kicks.

4 Creating the Model

i. Superimposition

The first step in creating the model is to superimpose the data on a grid. I decided to keep all the data points above the horizontal axis to ensure that none of the points are negative. It would not make sense for my ankle to have a 'negative position.' Before superimposing the points, I used Adobe Photoshop (Adobe, 2024) to remove the background and to colour the points in black. The background is now transparent which is useful so I can see the gridlines behind the image.

Figure 4. Removed background of data points (Author's Own, 2025)

After removing the background, I superimposed the data on Desmos, a virtual graphing calculator, shown on Figure 5 below.

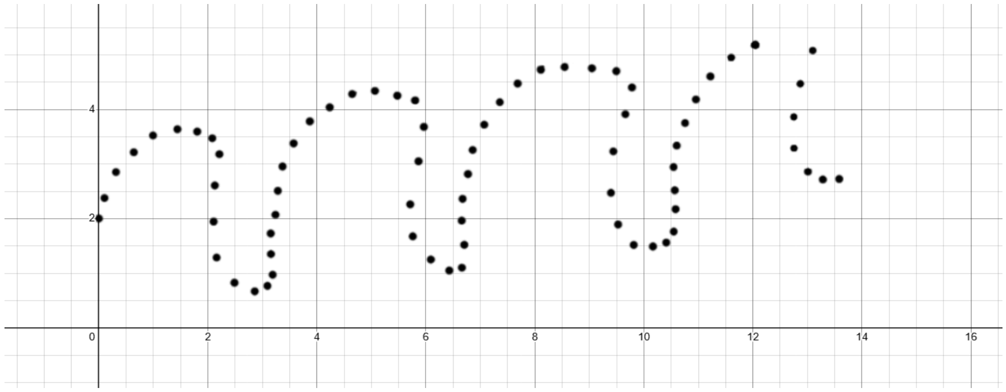


Figure 5. Superimposed graph on a coordinate plane (Author's Own, 2025)

Using desmos, I converted the dots from the image to points on a table. This allows for a more accurate representation of the data, as when zooming in to the original image, the 'feather' (soft edges of the points) spread too wide. Figure 6 below shows a large point area of a single dot.

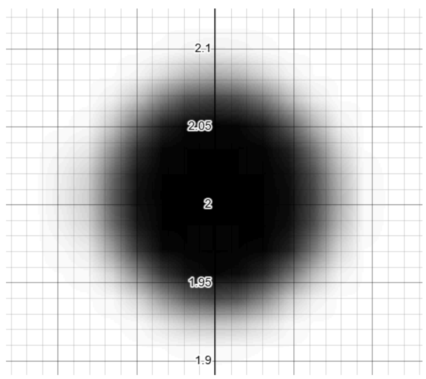


Figure 6. Inaccuracy of feathering (Author's Own, 2025)

After converting the dots to discrete tabular points, the shape of the dolphin kicks now look as shown on Figure 7 below.

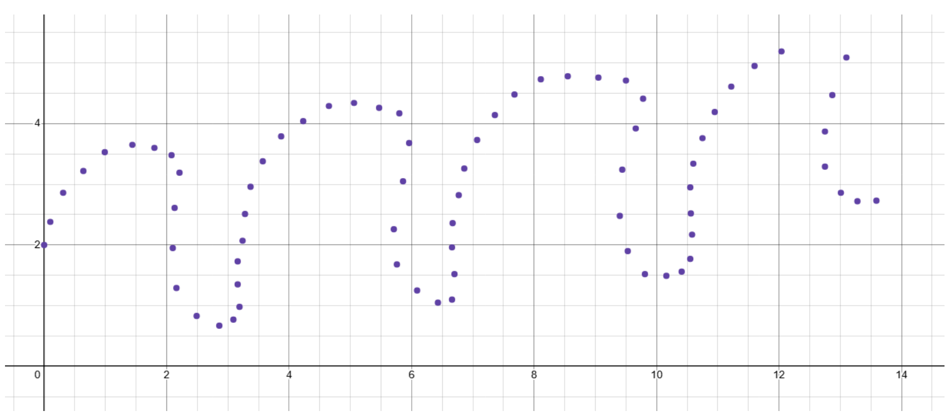


Figure 7. Desmos-plotted shape of dolphin kicks (Author's Own, 2025)

ii. Graphing the Shape

Now that the points have been superimposed, a graph can now be created that models the shape of the dolphin kicks. The initial step of plotting every point would be useful as it provides the exact x and y coordinates of each point, making it easier to find the specific function that matches their shape. The full table of points is shown in **Appendix 1.1**.

The first step in modelling this shape would be to split it into multiple different parts. This is because it is impossible to create an accurate model of this graph with simply one function, and more importantly, this currently is a function. Of course, it would be viable to model this using 72 different linear functions, however, for this exploration I am going to try and minimize the use of linear functions, only using them when deemed necessary.

Method 1) Parabolic Function

The first function that will be used to model the shape is a parabolic function. These functions are useful due to the fact that they can be written in vertex form. That way, I can choose any point to be my vertex and plug in another point to find its 'stretch.' These functions are in the form:

$$f(x) = a(x - h)^2 + k$$

where a is the horizontal stretch, and (h, k) are the coordinates of the vertex. The following points were modelled with a parabola due to its clear 'vertex point' and the stretch can be modified easily.

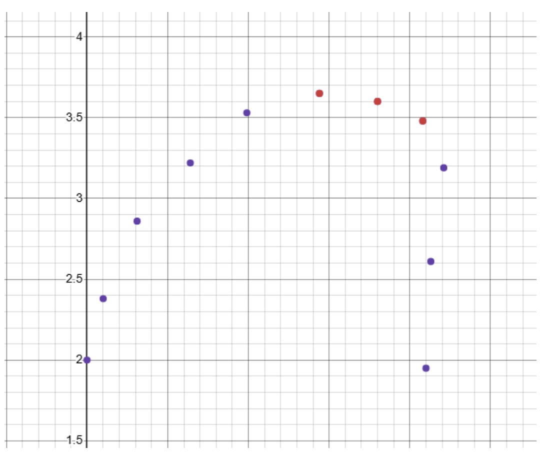


Figure 8. Parabolic method (in red)

Thanks to the points already being on desmos, the coordinates of the vertex are already known. In this case, the coordinates are $(1.44, 3.65)$. This can be plugged in to the vertex equation to get this function:

$$f(x) = a(x - 1.44)^2 + 3.65$$

Since I want this function to pass through the point $(2.08, 3.48)$, this value can be substituted for x and $f(x)$, yielding this result:

$$3.48 = a(2.08 - 1.44)^2 + 3.65$$

Rearranging to find a,

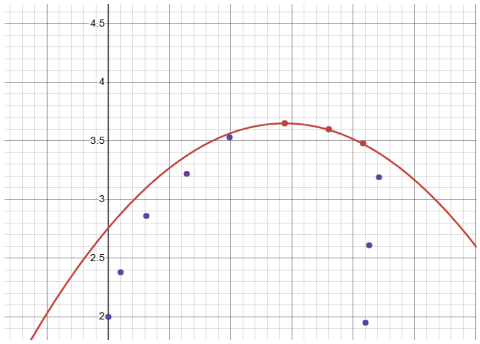
$$3.48 = a(0.64)^2 + 3.65$$

$$-0.17 = a(0.64)^2$$

$$-0.415 = a \text{ (3 s.f.)}$$

Therefore, the function is:

$$f(x) = -0.415(x - 1.44)^2 + 3.65$$



However, plugging this function straight into Desmos yields this result, and so a domain restriction needs to be set. It is clear that the function passes through three points (the ones in red), and so I will set the domain such that x lies in between the x values of the three points, yielding this function:

$$f(x) = -0.415(x - 1.44)^2 + 3.65 \{1.44 \leq x \leq 2.08\}$$

Figure 9. Parabolic function on Desmos

The domain restriction will be a technique used for all methods. The function now obtained yields the following:

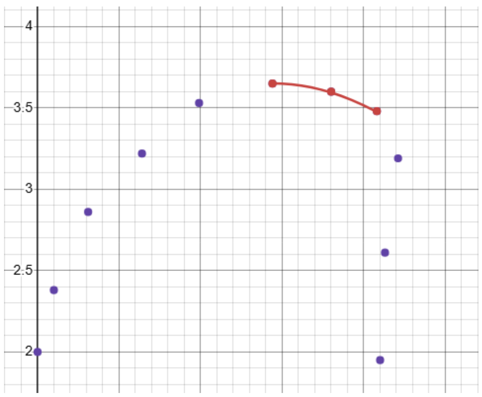
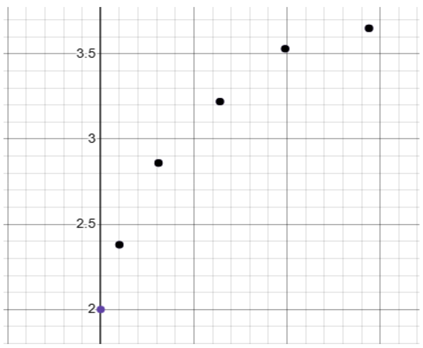


Figure 10. Parabola with domain restriction (Author's Own, 2025)

Method 2) Cubic Function (Using Online Regression)



The next function that will be examined is a cubic function in the form:

$$f(x) = ax^3 + bx^2 + cx + d$$

An online regression method will be used to find a cubic function in this form. For this specific example, a cubic function was chosen because the points somewhat plateau near the end, symbolizing a point of inflection.

Figure 11. Cubic method (in black)

x_2	y_2
0.31	2.86
0.64	3.22
0.99	3.53
1.44	3.65
0.1	2.38

Figure 13. Table of data (Author's Own, 2025)

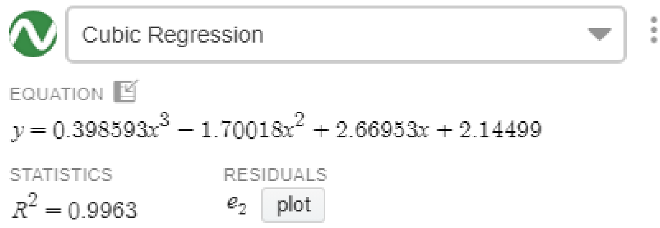


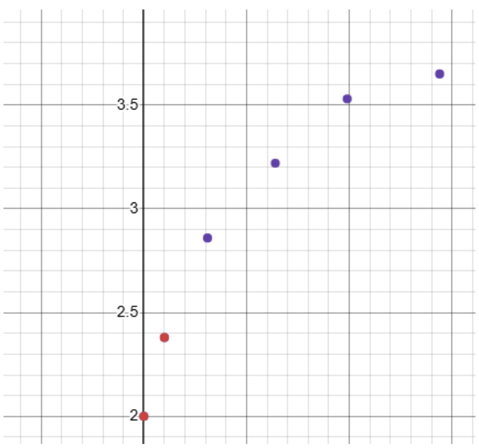
Figure 12. Regression model (Author's Own, 2025)

To proceed with the regression, the points were plotted on a separate table (to the left) and the 'regression' model was selected on Desmos (above).

Thus, the final function obtained is:

$$f(x) = 0.39x^3 - 1.70x^2 + 2.67x + 2.14 \{0.1 \le x \le 1.44\}$$

Method 3) Inverse Parabolic Function



The inverse parabolic function is only used once, exclusive to the very start of the dolphin kick shape. This was chosen as I imagined the first point at (0, 2) being the vertex of a parabola that lays on the y-axis. Conversely, a square root function was not chosen as its asymptote at $x = 0$ requires it to need both vertical and horizontal translations along with a stretch to match this shape.

Figure 14. Inverse parabolic points (in red)

The function will be in the form:

$$f(y) = a(y - h)^2 + k$$

where (k, h) are the coordinates of the vertex. In this case, (k, h) become $(0, 2)$

$$f(y) = a(y - 2)^2$$

To find a, the point $(0.1, 2.38)$ will be substituted in for y and $f(y)$.

$$0.1 = a(2.38 - 2)^2$$

$$0.1 = a(0.38)^2$$

$$a = 0.692 \text{ (3 s.f.)}$$

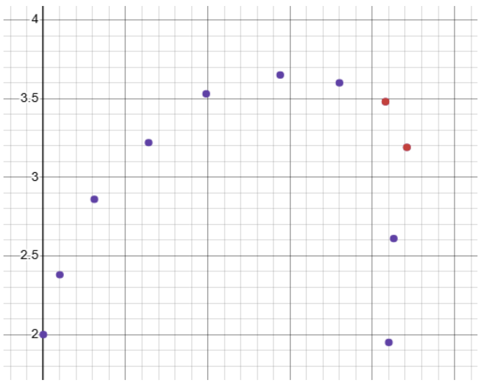
Therefore, the function now becomes:

$$f(y) = 0.692(y - 2)^2$$

Restricting the domain, the final function is:

$$f(y) = 0.692(y - 2)^2 \{2 \leq y \leq 2.38\}$$

Method 4) Linear Function



This method is primarily used to join two points together whose difference in x values are very small. The linear function is especially useful during the steep downwards motion of the kick, shown in red on the right. It would also be useful in modelling parts of the shape where the Δx (change in x) values between two points are negative.

Figure 15. Linear method (in red)

These functions are modelled using the point-slope formula, given by:

$$f(x) = m(x - x_1) + y_1$$

where the slope, m is given by,

$$m = \frac{y - y_1}{x - x_1}$$

In this specific example, the two points are $(2.08, 3.48)$ and $(2.21, 3.19)$, so the slope is:

$$m = \frac{3.19 - 3.48}{2.21 - 2.08}$$

$$= -2.23$$

Now, the equation is:

$$f(x) = -2.23(x - x_1) + y_1$$

Substituting the points $(2.08, 3.48)$ into x_1 and y_1 respectively,

$$f(x) = -2.23(x - 2.08) + 3.48$$

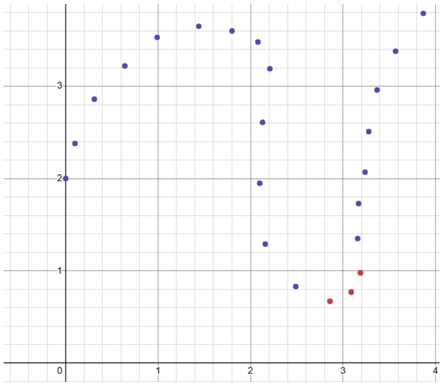
$$f(x) = -2.23x + 4.64 + 3.48$$

$$f(x) = -2.23x + 8.12$$

This result is the final function for this specific example is:

$$f(x) = -2.23x + 8.12 \{2.08 \leq x \leq 2.21\}$$

Method 5) Exponential Function



The next function that is used is the exponential function. These functions have asymptotes, and so they would work well when a series of points appears to be increasing exponentially while also having a clear point in which it flattens out and becomes asymptotic. This is seen in the figure on the left.

Figure 16. Exponential method (in red)

Exponential functions are in the form of:

$$f(x) = A^{(x-d)} + B$$

For this function specifically, I wanted the asymptote to be at the lowest point to avoid dealing with any translational complications. However, this would also need a very large base, as a steep growth would be needed. For modelling this part, I decided to test out different powers of 10 as the base. The higher the order of magnitude, the faster it would exponentiate. After trying orders of magnitude up to 10^6, I settled with $100,000$. The variable B represents the asymptote, which I wanted to be the lowest point. The y-value of this point is 0.67. Our function is now:

$$f(x) = 100000^{(x-d)} + 0.67$$

To find the horizontal translation, d, the point $(3.09, 0.77)$ will be substituted to calculate it.

$$0.77 = 100000^{(3.09-d)} + 0.67$$

$$0.1 = 100000^{(3.09-d)}$$

$$10^{-1} = 10^{5(3.09-d)}$$

$$-1 = 15.45 - 5d$$

$$d = 3.29$$

The final function, with the domain restriction, is:

$$f(x) = 100000^{(x-3.29)} + 0.67 \{2.89 \leq x \leq 3.19\}$$

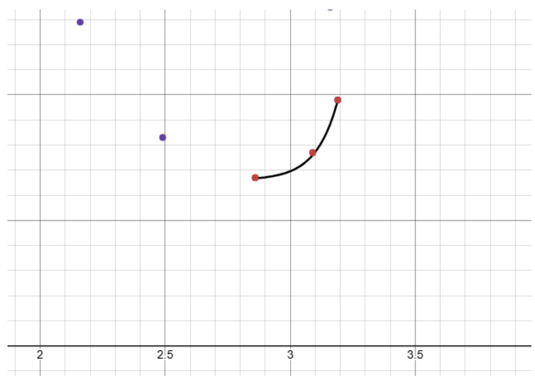
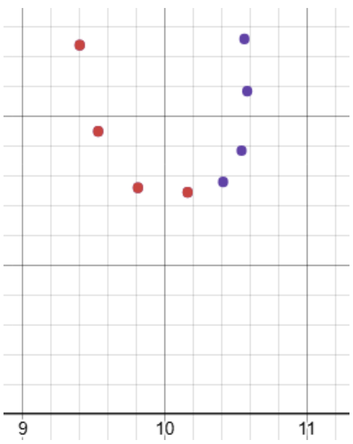


Figure 17. Exponential function plotted on Desmos (Author's Own, 2025)

Method 6) Quartic Function



The next function that is used is the quartic function. Similar to the parabolic function, this function is able to have one main vertex point. For the function to have one vertex, it must be written in the form:

$$f(x) \ = \ a(x - h)^4 + k$$

The reason the quartic function was used in certain cases as its vertex is flatter than the parabola's, useful for cases such as in Figure 18.

Figure 18. Quartic Method (in red)

In this case, the vertex is located at $(10.16, 1.49)$, and so these points will be substituted in.

$$f(x) \ = \ a(x - 10.16)^4 + 1.49$$

I want the function to pass through the point $(9.4, 2.48)$ and so these points will be substituted in to x and $f(x)$ to find a. Since a similar calculation was already done previously with the parabola, it will not be repeated. The final function is:

$$f(x) \ = \ 2.96(x - 10.16)^4 + 1.49 \, \{9.4 \leq x \leq 10.16\}$$

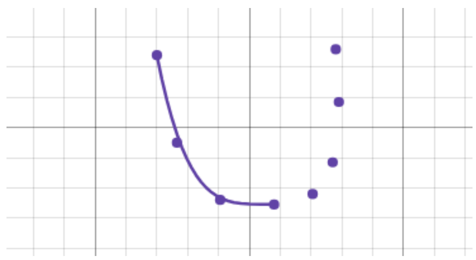
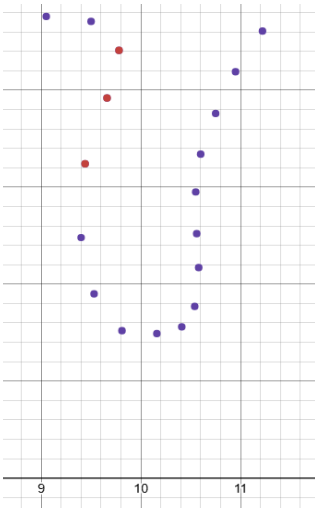


Figure 19. Quartic function on Desmos (Author's Own, 2025)

Method 7) Cubic Function (Without Online Regression)



The final method is to examine the cubic function once more but to not use an online regression function. The red highlighted points reminded me of a cubic function, however the regular $a(x - b)^3$ function has a stationary point of inflection. This calls for the need to have a function formed such as:

$$f(x) = (x - b)^3 + c(x - b) + d$$

This function now has a non-stationary point of inflection that gets steeper with increasing values of c, which is what is needed for this case. A cubic function of this form will have its point of inflection at a certain point, which in this case is given by (b, d).

Figure 20. Cubic method (in red)

I notice that the three points highlighted in red remind me of where the point of inflection is located, because there is a noticeable change in concavity. However, I cannot set the point of inflection to be at the point $(9.66, 3.92)$, in between the top and bottom ones, because it is not in the center. Therefore, I decided to take the midpoint of the top and bottom points and have the point of inflection located there. Using the midpoint formula,

$$M = (\frac{x+x_1}{2}, \frac{y+y_1}{2})$$

The points (x, y) and (x_1, y_1) are given by $(9.78, 4.41)$ and $(9.44, 3.24)$ respectively (though they are interchangeable due to the commutative property). Thus, the midpoint is located at:

$$M = (\frac{9.78+9.44}{2}, \frac{4.41+3.24}{2})$$

$$M = (9.61, 3.83)$$

Substituting these values for (b, d) respectively,

$$f(x) = (x - 9.61)^3 + c(x - 9.61) + 3.83$$

To find c, the function can be plotted on Desmos and a 'slider' can be created for c to find the best fit.

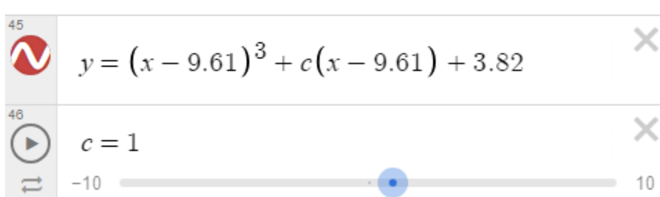


Figure 21. Desmos slider for c (Author's Own, 2025)

The slider was manipulated to find a steeper value for c, and thus the value $c = 3.3$ was obtained as it passes through all three points. Thus, the final function with the restricted domain is:

$$f(x) = (x - 9.61)^3 + 3.3(x - 9.61) + 3.83 \{9.44 \leq x \leq 9.78\}$$

All the methods listed above were used in combination to model the shape of the dolphin kick in full. For the sake of clarity and to avoid repetition, the calculations for every single part of the dolphin kick will not be listed, as the same techniques were used for all of them.

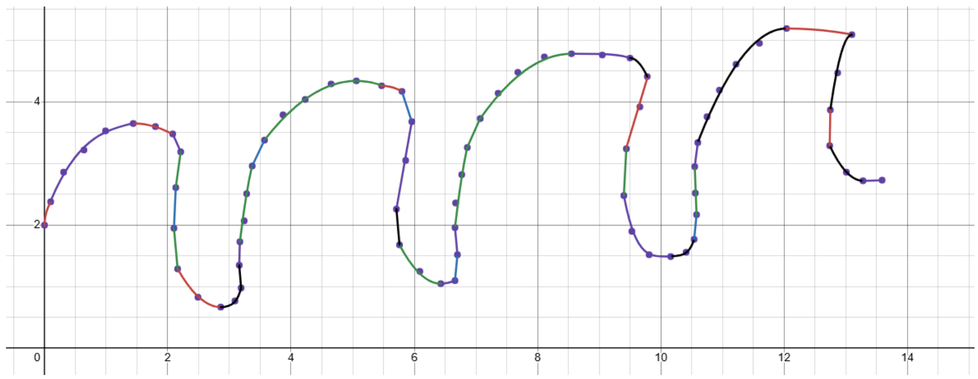


Figure 22. All functions used to model the dolphin kick (with data points) (Author's Own, 2025)

Without the data points, the final function is shown on Figure 23 below.

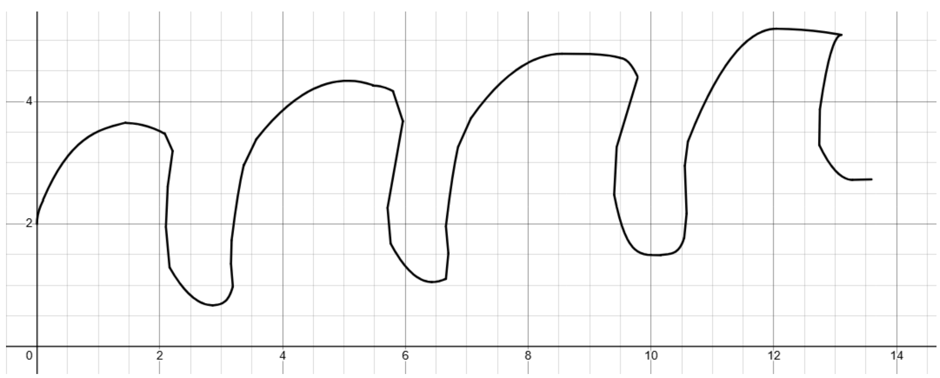


Figure 23. Final piecewise function in all black (without data points) (Author's Own, 2025)

The piecewise function is a result of 39 smaller functions. The full list of functions can be found in Appendix **1.2**. The screenshots were taken from Desmos.

5 Optimization

Interestingly enough, the final model for the shape of the dolphin kick does not use any sinusoidal functions. This goes against the established hypothesis of how the dolphin kick's shape can be generalized sinusoidally. However, this generalization can be made for the next part of this exploration, which is to find out if it would be possible to algebraically optimize the amplitude and frequency of my dolphin kicks. For this, a lot of assumptions need to be made about their nature.

The first being to assume they have an average amplitude and frequency. This will be done by creating this sinusoidal function:

$$f(x) = asin(bx) + g(x)$$

where a is the amplitude and b is the period. In this form, it is often assumed that the function $g(x)$ is a straight line given by $g(x) = c$. However, it is clear that the shape of the dolphin kicks increases by a linear factor. Thus, the function $g(x) = c$ must be changed to this form:

$$g(x) = mx + c$$

Substituting this result back into $f(x)$, we get:

$$f(x) = asin(bx) + mx + c$$

To find the slope of the linear factor, a line can be drawn through the shape that connects the point at the intercept $(0, 2)$ to a midpoint near the end of it. The point $(12.75, 3.87)$ was chosen.

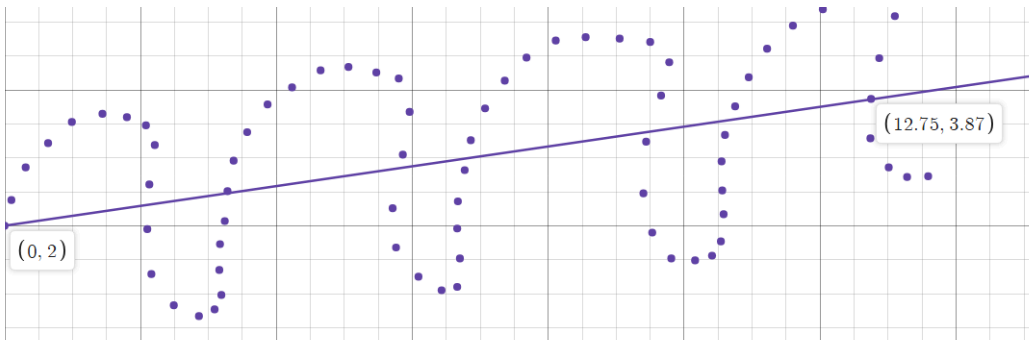


Figure 24. Line drawn through the dolphin kick shape. (Author's Own, 2025)

From here, the slope can be calculated. Using the points $(0, 2)$ and $(12.75, 3.87)$, the slope obtained is $m = 0.147$. It is also clear that the y-intercept of this line is located at $(0, 2)$, and so the value for c will be 2. Plugging these results into $f(x)$, the result is now:

$$f(x) = asin(bx) + 0.147x + 2$$

The next step is to find a, the average amplitude of the sine wave. In the shape, there are four noticeable peaks. To find the average amplitude of the sine wave, I simply took the point of the crest and their corresponding trough, and found the difference in their y-values. This also accounts for the slant given by the linear, or oblique axis. The difference was then added up and divided by 4 to find the amplitude. The calculations for the first peak will be shown. The peak is located at $(1.44, 3.65)$ and the trough is located at $(2.86, 0.67)$. The difference in y-values is thus given by:

$$d = 3.65 - 0.67$$
$$= 2.98$$

The differences for the next three peaks are 3.29, 3.29 again, and 2.47 respectively. Then, the average of these four peaks were found.

$$A = \frac{1}{4}(2.98 + 3.29 + 3.29 + 2.47)$$
$$= 3.0075$$
$$= 3.00 \text{ (3 s.f.)}$$

This value is the difference between the lowest and highest point, however we are looking for the difference between the linear axis. The amplitude would need to be divided in half.

$$a = 1.50$$

The function $f(x)$ now becomes,

$$f(x) = 1.5sin(bx) + 0.147x + 2$$

To find b, the general frequency, another slider was created in Desmos and adjusted to find the best frequency fit.

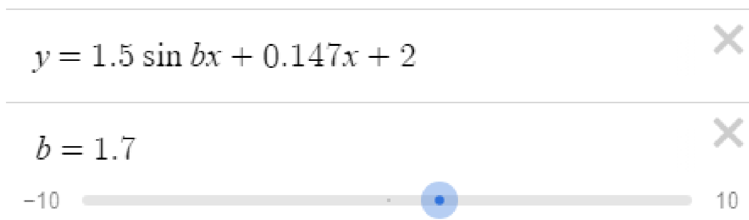


Figure 25. Slider to find the frequency of the dolphin kicks. (Author's Own, 2025)

The value $b = 1.7$ was obtained, and so the final general sinusoidal function looks like:

$$f(x) = 1.5sin(1.7x) + 0.147x + 2$$

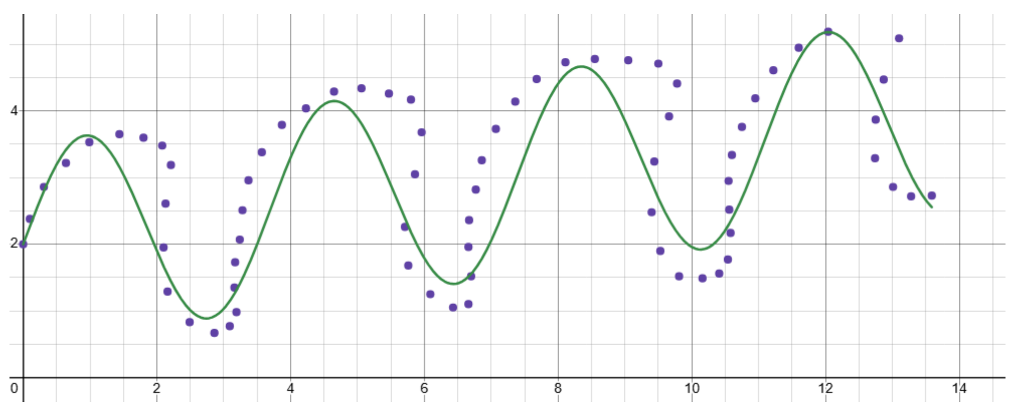


Figure 26. General sinusoidal function of the dolphin kicks (Author's Own, 2025)

Before trying to optimize the amplitude and frequency of my dolphin kicks, it is important to note what they manifest in the real world. As per the frequency, the value 1.7 signifies that I kick 1.7 times every second. This makes sense, since I was kicking for 2.4 seconds. The total amount of kicks is then $1.7 \times 2.4 = 4.08$ kicks, which is what is shown in the diagram (roughly four kicks). The amplitude is a bit harder to grasp. To find the amplitude (in cm) of the dolphin kicks, I used my foot as a reference.

I measured the distance between the instep of my foot (while flexed) and the bottom of my heel. This distance measured up to be exactly 12 cm. Going back to the original motion tracking video, I noticed that this distance was equal to the difference between the first two motion tracked points (as the distance between the bottom of my heel and the instep fit perfectly within the gap). The difference in y values between the first and second motion tracked points with coordinates $(0, 2)$ and $(0.1, 2.38)$ respectively is 0.38. This means that a coordinate distance of 0.38 corresponds to a distance of 12cm in the real world. The amplitude has a coordinate distance of 1.5. We can use ratios to figure out its distance in centimeters.

$$\frac{12cm}{0.38} = \frac{x}{1.5}$$

$$x = 47.4\text{cm}$$

Thus, the amplitude of my dolphin kicks is 47.4cm, meaning the total distance my foot travels is 94.8cm, roughly one metre, which makes physical sense. With this in mind, an attempt can now be made to algebraically optimize the frequency and amplitude of my dolphin kicks.

Preliminaries for Optimization

The first thing to do is to set up an equation. In order to set up my optimization for a and b, I have to define a function with velocity v as its parameter, which balances thrust (the forward force) and drag (the opposing force). Since thrust propels us forward and drag propels us backwards, their vector arrows go in opposite directions. I need to balance **both** thrust and drag as both factors are important. If I only maximize thrust, the velocity would increase, however it would also increase drag by a quadratic factor, as drag increases proportionally to the square of velocity. This means drag will grow too large and I would end up wasting energy. From this, we can derive a function:

$$f(v) = Thrust - Drag$$

The drag force equation is given by (Benson, n.d.):

$$F_D = \frac{1}{2}\rho v^2 C_D A$$

where v represents the velocity of the swimmer, C_D represents the drag coefficient, A represents the total cross sectional area of motion, and ρ represents the density of water

Now, the force of thrust has a general equation (NASA, 2022),

$$F_T = v\frac{dm}{dt}$$

where v is velocity and $\frac{dm}{dt}$ is the derivative of mass with respect to time.

Putting these two together, the equation now becomes:

$$f(v) = v\frac{dm}{dt} - \frac{1}{2}\rho v^2 C_D A$$

Gathering Data for Optimization

The next step is to find the coefficients, $\frac{dm}{dt}$, C_D and A. The mass of water being displaced has a density ρ and a volume V. Rearranging the general formula for density, we can make m the subject of the equation:

$$m = \rho V$$

Taking the derivative with respect to time, given that the density of water is a constant,

$$\frac{dm}{dt} = \rho\frac{dV}{dt}$$

$\frac{dV}{dt}$ is the volume flow rate of the water, or the volume of water being displaced per unit time. This depends on the cross sectional area of the swimmer, which we will call $a_{swimmer}$, and the velocity of the water being displaced. The first assumption we would need to make is that the velocity of the water being displaced is equal to the velocity of the swimmer at that point in time. Thus, the equation can be formed:

$$\frac{dV}{dt} = va_{swimmer}$$

An ideal condition that we need to assume is that the velocity of the swimmer at any given point is equal to the average velocity. This would allow me to use the average velocity formula:

$$\bar{v} = \frac{\Delta x}{\Delta t}$$

where Δx is the displacement (in cm), and Δt is the change in time

To find the velocity in one second, 30 frames (points) will be counted on the graph and its coordinate x-displacement will be found. This is because the video is recorded at 29.97 frames per second (fps), so one second will be approximately 30 frames. Since I am travelling mostly horizontally I want to focus on my horizontal movement. Using ratios, the displacement will be found in cm. In 30 frames, the coordinate x-displacement from the origin is 5.86, or around 185 centimetres. Therefore, the velocity is 185 centimetres per second, or 1.85 metres per second.

My cross sectional area will be estimated to be the shape of a rectangle, as to get an accurate area, my body would need to be further modelled, which would dilute this investigation. To find $a_{swimmer}$, it is important to consider drag. Since my head is the leading edge during streamline swimming, it will be taking most of the drag impact. The length will be equal to the distance between the back of head and the tip of my nose and the width will be the horizontal distance between my two shoulders (while in a streamlined position). I measured the distance between the back of my head and the tip of my nose to the best of my abilities and this turned out to be approximately 21 centimetres. I then have to measure the width of my shoulders, which is 42cm. Thus, the cross-sectional area is the product of these two measurements.

$$a_{swimmer} = 42 \times 21$$

$$= 882cm^2$$

Thus, we can substitute these values in to find $\frac{dV}{dt}$.

$$\frac{dV}{dt} = (185)(882)$$

$$= 163170cm^3s^{-1}$$

In cubic metres, the rate of flow of water is $0.163m^3s^{-1}$, and this can be substituted into the formula to find the rate of flow of mass.

$$\frac{dm}{dt} = \rho\frac{dV}{dt}$$

Water has a density of 1000 kilograms per cubic metre (Water Science School, 2018), so:

$$\frac{dm}{dt} = (1000)(0.163)$$

$$= 163kgs^{-1}$$

We can substitute this result back in to $f(v)$ and move on to the second term.

$$f(v) = 163v - \frac{1}{2}\rho v^2C_DA$$

The next step is to estimate the drag coefficient. This constant would usually need to be calculated using the force of drag, but since I do not have that, research can be conducted to find an estimate. The drag coefficient of a swimmer usually refers to the measure of efficiency of swimmer's technique, and generally ranges from $0.5 - 2$. When a swimmer is in a streamlined position, such as when performing underwater dolphin kicks, the value can be estimated to be in the lower range (roughly 0.7) (Novais et al., 2012). Therefore, 0.7 will be used for the drag coefficient.

The cross sectional area of the swimmer, A was found previously with the calculations of $a_{swimmer}$, so the value of $882cm^2$ or $0.0882m^2$ can simply be reused.

All these values can now be substituted into the function $f(v)$.

$$f(v) = 163v - \frac{1}{2}(1000)v^2(0.7)(0.0882)$$

$$f(v) = 163v - 500v^2(0.7)(0.0882)$$

$$f(v) = 163v - 30.9v^2$$

Optimization

To maximize velocity, the derivative of f with respect to v needs to be found. This is given by:

$$\frac{df}{dv} = 163 - 61.8v$$

The derivative will be set equal to 0 to find its critical points. The original function is a parabola that opens downwards (due to its negative v^2 coefficient), and so its critical point would be a maximum.

$$0 = 163 - 61.8v$$

$$61.8v = 163$$

$$v \approx 2.64ms^{-1} \text{ (3 s.f.)}$$

This means that the maximum velocity I can achieve is 2.64 metres per second. This is different to my current velocity, which is 1.85 metres per second. If we assume that the amplitude a and frequency b of the kicks are proportional to velocity, we can arrive at this result:

$$ab = 2.64$$

$$a = \frac{1}{b}(2.64)$$

This means that for every given amplitude, it can be substituted into this equation to find its corresponding maximum frequency. If we set a to be 1.5, in the model's case, the value for b would be 1.76. This means to achieve maximum velocity I would need to kick at roughly the same speed I already am kicking. This does not make physical sense, and so this result will be discussed in the Evaluation and Limitations section. This is the final optimization result for this exploration.

6 Conclusion

This investigation aimed to answer the following research question: *How can the shape of a swimmer's dolphin kick be modeled and optimized?*

The answer to the first part of the question is quite simple. Throughout this mathematical exploration, the dolphin kick of a swimmer was able to be modelled using a variety of different functions. To do this, footage of my own dolphin kicks were recorded and analyzed using Adobe After Effects to produce motion tracked points. These points were superimposed on a grid on Desmos and thus plotted. The model of the dolphin kick's shape was created using linear, quadratic, cubic, quartic and exponential functions. It was also generalized into a sinusoidal function given by:

$$f(x) = 1.5sin(1.7x) + 0.147x + 2$$

The second part of the question is a little bit more complicated. A relationship was established to optimize velocity and thus the amplitude and frequency of my dolphin kicks. The maximum velocity achieved was 2.64 metres per second. This result was achieved by forming an optimization function with velocity as its parameter, $f(v)$. The coefficients were found and the derivative of the function was taken to find its critical points. However, substituting a and b to maximize amplitude and frequency, the result does not quite make physical sense. In conclusion, the shape of a swimmer's dolphin kick is able to be modeled and an attempt to optimize it was made.

7 Evaluation, Extensions and Limitations

To address the fact that the optimal values for a and b did not really make physical sense, we can attribute that to the assumption that velocity is directly proportional to it. In reality, there would be some constant of proportionality, k, which would have needed to be determined experimentally. More limitations when it came to the optimization was the lack of information. I was not able to determine the drag coefficient experimentally either so it had to be researched. Additionally, only the horizontal portion of the velocity was considered, however in reality I am travelling in a diagonal direction. This brings up a possible extension, as I could use vectors to optimize my path of travel in the diagonal direction (using horizontal and vertical components). The velocity was, in reality, an overestimate, as a velocity of 2. 64 is actually very fast and that might be because of the slight inaccuracy of the coordinate-plane to distance (in real life) conversion.

A limitation could also be attributed to the recording device and software. It would also have been helpful to use computational fluid dynamic (CFD) simulation to model the drag and flow of water around my body. Simulations would also provide a more accurate value for the cross-sectional area of my body, rather than having to make assumptions.

When looking at creating the model itself, there are many extensions that could be made to model the kick shape more dynamically. More complex functions such as inverse trigonometric functions could possibly model it more accurately. The Lagrange polynomial interpolation could also have been investigated to extend this exploration further. It also would have been helpful to investigate multiple different techniques to model one section and compare them to see which one would fit the shape best. Regardless, there were many strengths with the piecewise model, allowing the shape to be broken down into many component parts.

There are also many environmental variables when it comes to the generalizability of my results and its implications on a wider scheme. Factors such as water turbulence, pool depth, and even swimmer body type varies from person to person and so the results would be different for everyone. The research question could be reframed to: *How can the shape of my dolphin kicks be modeled and optimized?* to account for how these results only apply to myself.

Overall, given the tools I had I believe this exploration made a great attempt at modelling and optimizing the shape of a swimmer's dolphin kick.

Appendices

Appendix 1.1

Table 1. All Data points (Author's Own, 2025)

x	y
0	2
0.1	2.38
0.31	2.86
0.64	3.22
0.99	3.53
1.44	3.65
1.8	3.6
2.08	3.48
2.21	3.19
2.13	2.61
2.1	1.95
2.16	1.29
2.49	0.83
2.86	0.67
3.09	0.77
3.19	0.98
3.16	1.35
3.16	1.73
3.24	2.07
3.28	2.51
3.37	2.96
3.57	3.38
3.87	3.79
4.23	4.04

x	y
4.65	4.29
5.06	4.34
5.47	4.26
5.8	4.17
5.96	3.68
5.86	3.05
5.71	2.26
5.76	1.68
6.09	1.25
6.43	1.05
6.66	1.1
6.7	1.52
6.66	1.96
6.67	2.36
6.77	2.82
6.86	3.26
7.07	3.73
7.36	4.14
7.68	4.48
8.11	4.73
8.55	4.78
9.05	4.76
9.5	4.71
9.78	4.41

x	y
9.66	3.92
9.44	3.24
9.4	2.48
9.53	1.9
9.81	1.52
10.16	1.49
10.41	1.56
10.55	1.77
10.58	2.17
10.56	2.52
10.55	2.95
10.6	3.34
10.75	3.76
10.95	4.19
11.22	4.61
11.6	4.95
12.04	5.19
13.1	5.09
12.87	4.47
12.75	3.87
12.75	3.29
13.01	2.86
13.28	2.72
13.59	2.73

Appendix 1.2

Below are screenshots of all the functions that were used to model the shape of my dolphin kicks (Author's Own, 2025)

3 $y = -0.415(x - 1.44)^2 + 3.65 \; \{1.44 \le x \le 2.08\}$

4 $x = 0.692(y - 2)^2 \{2 \le y \le 2.38\}$

5 $y = 0.399x^3 - 1.70x^2 + 2.67x + 2.14 \{0.1 \le x \le 1.44\}$

6 $y = -0.432(x - 5.06)^2 + 4.34 \{3.57 \le x \le 5.47\}$

7 $y = 38x - 118.73 \{3.16 \le x \le 3.17\}$

8 $y = -10.31(x - 3.57)^2 + 3.38 \{3.17 \le x \le 3.37\}$

9 $y = -\frac{37}{3}x + 40.32 \{3.16 \le x \le 3.19\}$

10 $y = 1.265(x - 2.86)^2 + 0.67 \{2.16 \le x \le 2.86\}$

11 $y = -0.479(x - 8.55)^2 + 4.78 \{7.07 \le x \le 8.55\}$

12 $y = -0.08(x - 8.55)^4 + 4.78 \{8.55 \le x \le 9.5\}$

13 $y = -0.89(x - 12.04)^2 + 5.19 \{10.6 \le x \le 12.04\}$

14 $y = 100000^{(x - 10.65)} + 1.49 \{10.16 \le x \le 10.54\}$

15 $y = 10x - 103.63\{10.54 \le x \le 10.58\}$

16 $y = -26x + 277.25\{10.55 \le x \le 10.58\}$

17 $y = 7.8x - 79.34\{10.55 \le x < 10.6\}$

18 $y = -0.088(x - 12.04)^2 + 5.19\{12.04 \le x \le 13.1\}$

19 $y = -10.52(x - 7.07)^2 + 3.73\{6.66 \le x \le 6.86\}$

20 $y = (x - 6.43)^2 + 1.05\{6.43 \le x \le 6.66\}$

21 $y = 10.5x - 68.83\{6.66 \le x < 6.7\}$

22 $y = 1.4(x - 6.43)^2 + 1.05\{5.76 \le x \le 6.43\}$

23 $y = -11x + 75.22\{6.66 \le x \le 6.7\}$

24 $y = 2.96(x - 10.16)^4 + 1.49\{9.4 \le x \le 10.16\}$

25 $y = 100000^{(x - 3.29)} + 0.67\{2.86 \le x < 3.19\}$

26 $y = 1.95(x - 13.28)^2 + 2.72\{12.74 \le x \le 13.28\}$

27 $y=-0.826(x-5.47)^2+4.26\{5.47\le x\le 5.8\}$

28 $y=2.1x-4.117\{3.37\le x\le 3.57\}$

29 $y=2.238x-12.09\{6.86\le x\le 7.07\}$

30 $y=0.032x+2.294\{13.28\le x\le 13.59\}$

31 $y=-9.95(x-13.1)^2+5.09\{12.75\le x\le 13.1\}$

32 $y=58.x-735.63\{12.74\le x\le 12.75\}$

33 $y=(x-9.61)^3+3.3(x-9.61)+3.82\{9.44\le x\le 9.78\}$

34 $y=19x-176.12\{9.4\le x\le 9.44\}$

35 $y=-2.23x+8.12\{2.08\le x\le 2.21\}$

36 $y=-3.82(x-9.5)^2+4.71\{9.5\le x\le 9.78\}$

37 $y=-3.063x+21.93\{5.8\le x\le 5.96\}$

38 $y=-11x+25.05\{2.1\le x\le 2.16\}$

39 $y=5.68x-30.17\{5.71\le x\le 5.96\}$

40 $y=-11.6x+68.496\{5.71\le x\le 5.76\}$

41 $y=22x-44.25\{2.1\le x\le 2.13\}$

42 $y=7.25x-12.83\{2.13\le x\le 2.21\}$

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